

Incarnations of Skyrmions

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Abstract

Skyrmions can be transformed into lumps or baby-Skyrmions by being trapped inside a domain wall. Here we find that they can also be transformed into sine-Gordon kinks when confined by vortices, resulting in confined Skyrmions. We show this both by an effective field theory approach and by direct numerical calculations. The existence of these trapped and confined Skyrmions does not rely on higher-derivative terms when the host solitons are flat or straight. We also construct a Skyrmion as a twisted vortex ring in a model with a sixth-order derivative term.

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I. INTRODUCTION

Topological solitons and instantons [1] play a significant role in diverse areas of physics such as quantum field theories, string theory, cosmology [2] and condensed matter systems [3]. For instance, Yang-Mills instantons play an especially important role in non-perturbative dynamics of quantum gauge-field theories. Relations between solitons and instantons with different dimensionalities are important for a unified understanding of these objects, which may explain or even reveal unexpected relations between field theories defined in different dimensions. Yang-Mills instantons in pure Yang-Mills theory in four-dimensional Euclidean space are particle-like topological solitons in $d = 4+1$ dimensional spacetime. When coupled to Higgs fields in the Higgs phase, they are unstable and shrink in the bulk. They can, however, stably live inside host solitons, in which they transform themselves to other kinds of solitons: They transform themselves into Skyrmons [4] when trapped inside a non-Abelian domain wall [5], or lumps [6] when inside a non-Abelian vortex [7, 8] and sine-Gordon (SG) kinks [9] when inside a monopole string, as summarized in (d), (e), and (f) in Tab. I. One such composites, namely a lump inside a non-Abelian vortex, elegantly explains the coincidence of mass spectra between field theories in different dimensions [10, 11]. If one compactifies the world volume of host solitons, instantons also can exist, not only as such trapped solitons; they become twisted closed domain walls, vortex sheets or monopole strings, when the moduli S^3 , S^2 , or S^1 of these host solitons are wound around their compact world volumes, also of the shape S^3 , S^2 , or S^1 , respectively [12] [see (d), (e), and (f) in Tab. I].

A similar relation is known in $d = 2+1$ dimensions, in which lumps [13] or baby-Skyrmions [14, 15] become sine-Gordon kinks [16–18] when trapped inside a $\mathbb{C}P^1$ domain wall [19, 20], corresponding to (a) in Tab. I. This relation was already known earlier in condensed-matter systems such as Josephson junctions of two superconductors [21], ferromagnets [22], and ^3He superfluids [3]. When one compactifies the world volume of the domain wall to S^1 , it becomes an isolated lump or baby-Skyrmion as a closed domain line with a twisted $U(1)$ modulus [17]. Similar twisted closed wall lines as vortices also exist in condensed-matter systems such as p-wave superconductors [23].

In this paper, we further pursue these relations between solitons with different dimensionalities, by focusing on Skyrmons [24] in $d = 3+1$ dimensions, which are characterized by the topological charge $\pi_3(S^3) \simeq \mathbb{Z}$, i.e. the baryon number. It was already found that Skyrmons

soliton /dim	π_n in bulk		host solitons	π_n of host	codim	moduli	w.v. shape	w.v. soliton	π_n on w.v.
lump 2+1 dim	$\pi_2(S^2)$	(a)	$\mathbb{C}P^1$ domain wall	π_0	1	S^1	\mathbb{R}^1 or S^1	SG kink	$\pi_1(S^1)$
Skyrmion 3+1 dim	$\pi_3(S^3)$	(b)	NA domain wall	π_0	1	S^2	\mathbb{R}^2 or S^2	lump	$\pi_2(S^2)$
		(c)	vortex string	π_1	2	S^1	\mathbb{R}^1 or S^1	SG kink	$\pi_1(S^1)$
Instanton 4+1 dim	$\pi_3(G)$	(d)	NA domain wall	π_0	1	S^3	\mathbb{R}^3 or S^3	Skyrmion	$\pi_3(S^3)$
		(e)	NA vortex sheet	π_1	2	S^2	\mathbb{R}^2 or S^2	lump	$\pi_2(S^2)$
		(f)	monopole string	π_2	3	S^1	\mathbb{R}^1 or S^1	SG kink	$\pi_1(S^1)$

TABLE I: Host solitons of trapped instantons (Skyrmions). (a), (d), (e) and (f) were already pointed out in Ref. [12]. The shape of the world-volume can be noncompact, \mathbb{R}^n , or compact, S^n , corresponding to trapped and untrapped instantons (Skyrmions), respectively. w.v. stands for “world volume,” NA denotes “non-Abelian” and G denotes the gauge group.

become baby-Skyrmions or lumps when trapped inside a non-Abelian domain wall [25–27], and a spherical domain wall with twisted S^2 moduli is a Skyrmion [28], both corresponding to (b) in Tab. I (this relation was generalized to N -dimensional Skyrmions becoming $N - 1$ dimensional Skyrmions inside non-Abelian domain walls in [26]). The last piece which was missing in Tab. I corresponds to (c), which we will work out explicitly in this paper. We consider a potential term motivated by two-component Bose-Einstein condensates (BECs) of ultracold atoms with repulsive interactions [29] (see Appendix A), which are known to admit a variety of topological solitons such as domain walls, vortices, Skyrmions (as vortons) [30–32] and D-brane solitons (vortices ending on a domain wall) [31, 33]. We denote this model the BEC Skyrme model. The model admits a global vortex solution with a $U(1)$ modulus. We then deform the potential by a perturbation which introduces a potential for the $U(1)$ modulus, and construct a stable Skyrmion as a sine-Gordon kink trapped inside the straight vortex. We achieve this by two approaches: the effective field theory on solitons [34] which is based on the moduli approximation [35] and by direct numerical computations. Together with the previous result, we have two kinds of solitons: a domain wall and a vortex, able to host Skyrmions, as illustrated in Fig. 1 (a) and (b).

We then compactify the world-volume of the vortex-line to S^1 for which we consider the

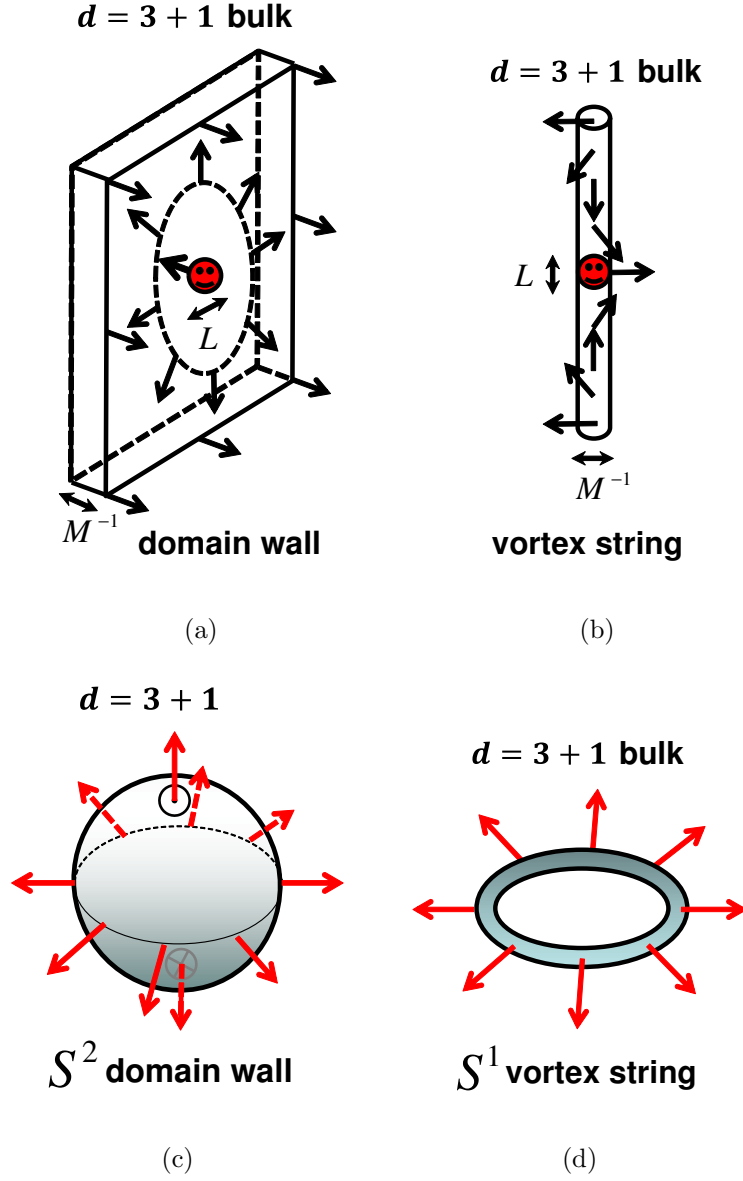


FIG. 1: Incarnation of Skyrmions. (a) A lump or a baby-Skyrmion on a flat domain wall, (b) A sine-Gordon kink on a straight vortex, (c) A spherical domain wall with twisted S^2 moduli, (d) A vortex ring with twisted S^1 modulus.

BEC potential without any perturbations. The configuration becomes a vortex ring with a twisted $U(1)$ modulus, that is, a vorton [36] (except for the time dependence which is usually required for vortons). In fact, it is known in the context of BECs that a Skyrmion is nothing but a vorton [30–32]. Together with the previous results of [27], we find two possible incarnations of Skyrmions; one corresponds to a Skyrmion as a spherical domain

wall with the S^2 moduli twisted along the S^2 world-volume [28], while the other corresponds to a closed vortex string with the $U(1)$ modulus twisted along the S^1 world-volume, as illustrated in Fig. 1 (c) and (d).

For a certain choice of perturbed potential which is described above, a Skyrmion is attached by the same kind of vortices from both of its sides. With a different choice of perturbed potential term, we also construct a (half-)Skyrmion attached by different kinds of vortices from both of its sides. In this case, we cannot compactify the world-volume since the vortices attached from both of its sides are different. The latter is similar to a confined monopole in the Higgs phase: In the Higgs phase, magnetic fluxes of a 't Hooft-Polyakov monopole [37] are squeezed to form vortices, and the monopole becomes a kink inside a vortex [11, 38, 39]. This was the first prime example of composite Bogomol'nyi-Prasad-Sommerfield (BPS) solitons, see Refs. [40–42] for a review. We call our configurations confined Skyrmions.

This paper is organized as follows. In Sec. II, we present the Skyrme models we consider in this paper. In particular, we use two different kinds of potential terms for model 1 and 2, respectively. Model 1 was already studied in our previous works, except for the lump-type of Skyrmion residing in the domain wall, while model 2 is introduced in this paper and is motivated by two-component BECs. In Sec. III, we construct domain wall solutions for model 1 and global vortex solutions for model 2, which will serve as host solitons for our baby solitons: baby Skyrmions (or lumps) and sine-Gordon kinks, respectively. In Sec. IV, we construct Skyrmions being baby-Skyrmions (or lumps) and sine-Gordon kinks in the effective theories of the domain wall and the vortex as host solitons in model 1 and model 2, respectively. In Sec. V, we provide full numerical solutions for such composite Skyrmions. We also construct a vortex ring with the twisted $U(1)$ modulus as a Skyrmion in model 2. Sec. VI is devoted to a summary and discussion. In Appendix A, we explain a potential term of two-component BECs and its relation to our model. In Appendix B, we show full numerical solutions of a half-Skyrmion trapped inside a vortex in model 2.

II. SKYRME-LIKE MODELS

We consider the $SU(2)$ principal chiral model or the Skyrme model with higher-derivative terms, in $d = 3 + 1$ dimensions. With the $SU(2)$ valued field $U(x) \in SU(2)$, the Lagrangian which we consider is given by

$$\mathcal{L} = \frac{f_\pi^2}{16} \text{tr} (\partial_\mu U^\dagger \partial^\mu U) + \mathcal{L}_4 + \mathcal{L}_6 - V(U), \quad (1)$$

with the Skyrme [24] and sixth-order derivative term, respectively

$$\mathcal{L}_4 = \frac{\kappa}{32e^2} \text{tr} ([U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2), \quad (2)$$

$$\mathcal{L}_6 = \frac{c_6}{36e^4 f_\pi^2} (\epsilon^{\mu\nu\rho\sigma} \text{tr} [U^\dagger \partial_\nu U U^\dagger \partial_\rho U U^\dagger \partial_\sigma U])^2. \quad (3)$$

The symmetry of the Lagrangian with $V = 0$ is $\tilde{G} = SU(2)_L \times SU(2)_R$ acting on U as $U \rightarrow U' = g_L U g_R^\dagger$. This is spontaneously broken to $\tilde{H} \simeq SU(2)_{L+R}$ acting as $U \rightarrow U' = g_{L+R} U g_{L+R}^\dagger$ so that the target space is $\tilde{G}/\tilde{H} \simeq SU(2)_{L-R}$. The conventional potential term is $V = m_\pi^2 \text{tr} (2\mathbf{1}_2 - U - U^\dagger)$, which breaks the symmetry \tilde{G} to $SU(2)_{L+R}$ *explicitly*.

In this paper, we consider both cases where the higher-derivative terms are turned off ($\kappa = c_6 = 0$) and where either the Skyrme term or the sixth-order term is turned on. A BPS model was discovered some years back [43], which consists of only the sixth-order term as well as appropriate potentials. This type of model is not in our parameter space here as it corresponds to $f_\pi \rightarrow 0$ (and $\kappa = 0$) which is not possible as we will rescale away f_π , see below. It is interesting, however, for phenomenological reasons due to the possibility of parametrically small (classically) binding energies and a large extended symmetry (volume-preserving diffeomorphisms). In this paper, we consider systems which perhaps are closer to condensed-matter systems and do have a kinetic term and large binding energies here are thus not an immediate concern.

If we rewrite the Lagrangian in terms of two complex scalar fields $\phi^T = \{\phi_1(x), \phi_2(x)\}$, defined by

$$U = \begin{pmatrix} \phi_1 & -\phi_2^* \\ \phi_2 & \phi_1^* \end{pmatrix}, \quad (4)$$

subject to the constraint

$$\det U = |\phi_1|^2 + |\phi_2|^2 = 1, \quad (5)$$

then the Lagrangian can be rewritten as

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2} \partial_\mu \phi^\dagger \partial^\mu \phi - \frac{\kappa}{4} \left[(\partial_\mu \phi^\dagger \partial^\mu \phi)^2 - \frac{1}{4} (\partial_\mu \phi^\dagger \partial_\nu \phi + \partial_\nu \phi^\dagger \partial_\mu \phi)^2 \right] \\
&\quad + \frac{c_6}{144} (\epsilon^{\mu\nu\rho\sigma} [\phi^\dagger (\partial_\nu \phi \partial_j \phi^\dagger + \sigma^2 \partial_i \phi^* \partial_\rho \phi^T \sigma^2) \partial_\sigma \phi + \text{c.c.}])^2 - V(\phi, \phi^*) \\
&= \frac{1}{2} \partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n} - \frac{\kappa}{4} [(\partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n})^2 - (\partial_\mu \mathbf{n} \cdot \partial_\nu \mathbf{n})^2] \\
&\quad + \frac{c_6}{36} (\epsilon^{\mu\nu\rho\sigma} \epsilon^{ABCD} \partial_\nu n_A \partial_\rho n_B \partial_\sigma n_C n_D)^2 - V(\mathbf{n}),
\end{aligned} \tag{6}$$

where we have rescaled the Lagrangian density such that energy is measured in units of $f_\pi/(2e)$ and length is measured in units of $2/(ef_\pi)$ and we have introduced the real four-vector scalar fields $\mathbf{n}(x) = \{n_A(x)\} = \{n_1(x), n_2(x), n_3(x), n_4(x)\}$ satisfying $\mathbf{n}^2 = \sum_A n_A^2 = 1$, defined by $\phi_1 = n_1 + in_2$ and $\phi_2 = n_3 + in_4$ ($A, B, C, D = 1, 2, 3, 4$).

The target space (the vacuum manifold with $V = 0$) $\mathcal{M} \simeq SU(2) \simeq S^3$ has a nontrivial homotopy group

$$\pi_3(\mathcal{M}) = \mathbb{Z}, \tag{7}$$

which admits Skyrmions as usual. The baryon number (Skyrme charge), $B \in \pi_3(S^3)$, is defined as

$$\begin{aligned}
B &= -\frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{tr} (U^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U) \\
&= \frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{tr} (U^\dagger \partial_i U \partial_j U^\dagger \partial_k U) \\
&= \frac{1}{24\pi^2} \int d^3x [\epsilon^{ijk} \phi^\dagger (\partial_i \phi \partial_j \phi^\dagger + \sigma^2 \partial_i \phi^* \partial_j \phi^T \sigma^2) \partial_k \phi + \text{c.c.}] \\
&= -\frac{1}{12\pi^2} \int d^3x \epsilon^{ABCD} \epsilon^{ijk} \partial_i n_A \partial_j n_B \partial_k n_C n_D \\
&= -\frac{1}{2\pi^2} \int d^3x \epsilon^{ABCD} \partial_1 n_A \partial_2 n_B \partial_3 n_C n_D.
\end{aligned} \tag{8}$$

Instead of the conventional potential term, we consider here potential terms of the form

$$V = V_1 + V_2, \tag{9}$$

where V_1 is the dominant potential and it admits a host soliton such as a domain wall or a vortex while V_2 is a subdominant potential admitting a soliton inside of the host soliton – a baby soliton. We consider two theories: model 1 admitting Skyrmions as baby-Skyrmions inside a domain wall; and model 2 admitting Skyrmions as sine-Gordon solitons inside a vortex.

For model 1, we take the potential to be

$$V_1 = \frac{1}{2}M^2(1 - n_4^2), \quad V_2 = -\frac{1}{2}m_3^2 n_3^{a_3} \quad (10)$$

with $a_3 = 1$ or 2 . When m_3 is zero, V_2 vanishes and the potential admits two discrete vacua: $n_4 = \pm 1$. This allows for a domain wall interpolating between the two vacua [44] as we show in the next section. With a nonzero $m_3 > 0$, there still remain two vacua and a domain wall interpolating between them as long as $m_3 < M$ [25–27].

For model 2, we take the potential to be

$$\begin{aligned} V_1 &= \frac{1}{8}M^2 \left[1 - (\phi^\dagger \sigma^3 \phi)^2 \right] = \frac{1}{2}M^2 |\phi_1|^2 |\phi_2|^2 = \frac{1}{2}M^2 (n_1^2 + n_2^2) (n_3^2 + n_4^2), \\ V_2 &= -\frac{1}{2}m_3^2 n_3^{a_3}, \end{aligned} \quad (11)$$

with $a_3 = 1$ or 2 . The potential V_1 is motivated by two-component BECs (see Appendix A), and admits global vortices [29] and Skyrmions as vortons [30–32].

III. DOMAIN WALLS AND VORTICES AS HOST SOLITONS

In this section, we construct a domain wall and a vortex for models 1 and 2, respectively, with their respective V_1 potentials in the limit $V_2 = 0$. We will take into account the effect of V_2 in the next sections.

A. Model 1: the domain wall

Model 1 has two discrete vacua and admits a domain wall solution interpolating between them. First, we consider only V_1 with $m_3 = 0$ ($V_2 = 0$). With the Ansatz $\mathbf{n} = \{0, 0, \sin f(x), \cos f(x)\}$ we have

$$\mathcal{L} = -\frac{1}{2}(\partial_x f)^2 - \frac{1}{2}M^2 \sin^2 f. \quad (12)$$

This Lagrangian density is the sine-Gordon model admitting a domain-wall solution

$$f = 2 \tan^{-1} \exp(\pm Mx). \quad (13)$$

The domain wall in this type of model was first studied in Ref. [44]. The most general solution is given by

$$\mathbf{n} = \{b_1 \sin f(x), b_2 \sin f(x), b_3 \sin f(x), \cos f(x)\}, \quad (14)$$

which has moduli in the form of a constant three-vector \mathbf{b} with unit length $\mathbf{b}^2 = 1$. These are Nambu-Goldstone (NG) modes due to the spontaneously broken $O(3)$ symmetry, which is broken down to $O(2)$ in the presence of the domain wall (13). These S^2 moduli of the domain wall were discussed in Refs. [25, 26, 45].

B. Model 2: the vortex

Model 2 allows for global vortices. The vortex of ϕ_1 traps ϕ_2 in its core and carries a $U(1)$ modulus being the phase of ϕ_2 . For constructing the vortex, we use the following Ansatz

$$\phi_1 = \sin f(r)e^{i\phi}, \quad \phi_2 = \cos f(r), \quad (15)$$

where r, ϕ are polar coordinates in the plane. This simplifies the Lagrangian density to

$$-\mathcal{L} = \frac{1}{2}f_r^2 + \frac{1}{2r^2}\sin^2 f + \frac{\kappa}{2r^2}\sin^2(f)f_r^2 + \frac{1}{8}M^2\sin^2(2f), \quad (16)$$

and the equation of motion reads

$$f_{rr} + \frac{1}{r}f_r - \frac{1}{2r^2}\sin 2f + \frac{\kappa}{r^2}\sin^2 f \left(f_{rr} - \frac{1}{r}f_r \right) + \frac{\kappa}{2r^2}\sin(2f)f_r^2 - \frac{1}{4}M^2\sin 4f = 0. \quad (17)$$

The boundary conditions for the vortex system are

$$f(0) = 0, \quad f(\infty) = \frac{\pi}{2}. \quad (18)$$

Numerical solutions are shown in Fig. 2 for $\kappa = 0, 1, \dots, 4$. Notice that when κ is turned on, the vortex widens up and the energy density at the origin drops significantly.

This vortex is a particular solution, while the most general solution is given by

$$\phi_1 = \sin f(r)e^{i\phi}, \quad \phi_2 = \cos f(r)e^{i\zeta}, \quad (19)$$

where $\mathbf{b} = \{\cos \zeta, \sin \zeta\}$ is a constant $U(1)$ modulus.

IV. EFFECTIVE THEORY APPROACH

In this section we will review the results of Ref. [34]. The main idea is to take a soliton solution and integrate out the soliton to obtain the effective theory for the moduli inhabiting the soliton in question. Here we will consider only the leading-order effective Lagrangian. For details and a discussion of the expansion, see Ref. [34]. We will next consider each model in turn.

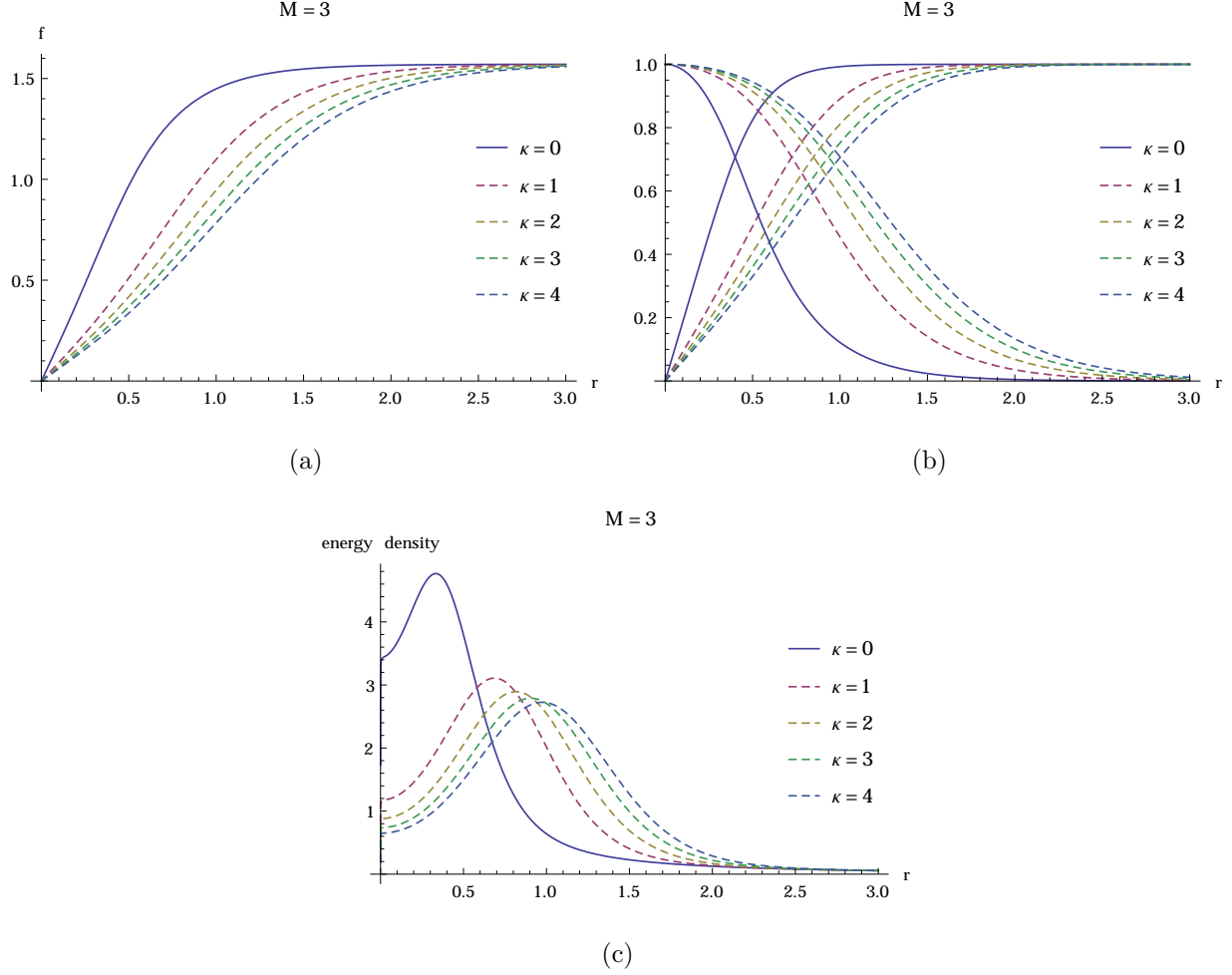


FIG. 2: (a) Vortex profile function f without the Skyrme term $\kappa = 0$ (blue solid curve) and with the Skyrme term $\kappa = 1, \dots, 4$ (dotted curves) for mass $M = 3$. (b) Condensate fields $|\phi_1| = \sin f$ and $|\phi_2| = \cos f$ (ϕ_1 vanishes at the origin). (c) Corresponding energy densities.

A. Model 1: Skyrmions as baby Skyrmions inside a domain wall

We start with the Lagrangian density (6) and integrate over the codimension of the domain wall. Here we will take just the leading-order effective Lagrangian and neglect the backreaction of the baby soliton to the domain wall. This is a good approximation when there is a separation of scales between the domain wall mass and the baby-soliton's typical scales. At leading-order the domain wall is flat which leads to a drastic simplification and the integration over its codimension yields

$$-\mathcal{L}^{\text{eff}} = \left(\frac{a_{2,0}}{M} + \kappa a_{2,2} M \right) (\partial_\alpha \mathbf{b})^2 + \left(\frac{\kappa a_{4,0}}{2M} + c_6 a_{4,2} M \right) (\partial_\alpha \mathbf{b} \times \partial_\beta \mathbf{b})^2 - \frac{a_{2,0} m_3^2}{M} b_3^{a_3}, \quad (20)$$

where M is the mass scale of the domain wall, $\alpha, \beta = t, y, z$, $a_3 = 1, 2$ and we have defined dimensionless constants as follows

$$a_{k,\ell} \equiv \frac{M}{2} \int dx \sin^k f \left(\frac{\partial_x f}{M} \right)^\ell = \frac{1}{2} \int d\xi \operatorname{sech}^{k+\ell} \xi = \frac{\sqrt{\pi} \Gamma\left(\frac{k+\ell}{2}\right)}{\Gamma\left(\frac{1+k+\ell}{2}\right)}, \quad (21)$$

where $\xi = Mx$ and the last two equalities have been evaluated using the flat domain wall (13) and the result is given in terms of the gamma function. Notice that the coefficients depend only on the sum of k and ℓ and so we can evaluate them as $a_{k,\ell} = a_{k+\ell}$:

$$a_2 = 1, \quad a_4 = \frac{2}{3}, \quad a_6 = \frac{8}{15}, \quad \dots \quad (22)$$

Inserting the coefficients we get

$$-\mathcal{L}^{\text{eff}} = \left(\frac{1}{M} + \frac{2}{3} \kappa M \right) (\partial_\alpha \mathbf{b})^2 + \left(\frac{\kappa}{3M} + \frac{8}{15} c_6 M \right) (\partial_\alpha \mathbf{b} \times \partial_\beta \mathbf{b})^2 - \frac{m_3^2}{M} b_3^{a_3}. \quad (23)$$

The existence of a baby-Skyrmion living on the domain wall requires a non-vanishing $\kappa > 0$ or $c_6 > 0$ as well as a non-zero m_3 . The size of the baby-Skyrmion can be estimated by a scaling argument [46] to be

$$\frac{1}{L} \sim \sqrt[4]{\frac{15m_3^2}{5\kappa + 8c_6 M^2}}. \quad (24)$$

A second kind of soliton which can inhabit the domain wall is the lump, which exists when $\kappa = c_6 = m_3 = 0$ and it will possess a size modulus [25].

The (full 3D) baryon charge is composed by the domain wall charge and the baby-Skyrmion charge and is given by

$$B = \frac{1}{\pi} \int d^3x \mathcal{Q} f_x = Q, \quad (25)$$

where we have used that there is only a single domain wall in this setup while

$$\mathcal{Q} = \frac{1}{8\pi} \epsilon^{ij} \mathbf{n} \cdot \partial_i \mathbf{n} \times \partial_j \mathbf{n}, \quad (26)$$

is the baby-Skyrmion charge density and Q the baby-Skyrmion number (charge).

B. Model 2: Skyrmions as kinks on a vortex

The next and final type of soliton we will consider is the vortex which has codimension two and a single world-volume direction. We again take the Lagrangian density (6) and

TABLE II: Coefficients for the effective Lagrangian density for sine-Gordon kinks living on a straight vortex for various values of κM^2 .

κM^2	0	1	2	3	4
$a_{2,0,0}$	0.5106	0.7224	0.8678	0.9866	1.090
$a_{2,2,0}$	0.1616	0.1550	0.1519	0.1499	0.1484
$a_{2,0,2}$	0.1745	0.1816	0.1852	0.1877	0.1896
$a_{2,2,2}$	0.06072	0.04172	0.03438	0.03007	0.02712

integrate over the two codimensions of the vortex to obtain

$$-\mathcal{L}^{\text{eff}} = \left[\frac{a_{2,0,0}}{M^2} + \kappa(a_{2,2,0} + a_{2,0,2}) + 2c_6 a_{2,2,2} M^2 \right] (\partial_\alpha \mathbf{b})^2 - \frac{a_{2,0,0} m_3^2}{M^2} b_1^2, \quad (27)$$

where M is the mass scale of the vortex, $\alpha = t, z$ and the dimensionless coefficients in the effective Lagrangian density read

$$a_{k,\ell,m} \equiv \pi M^{2-\ell-m} \int dr r^{1-\ell} \cos^k f \sin^\ell f (f_r)^m. \quad (28)$$

This effective theory possesses sine-Gordon kinks. Hence, the vortex can bear sine-Gordon kinks in terms of its twisted S^1 modulus and each of these kinks correspond to Skyrmsions in the full 3-dimensional theory.

Unfortunately, the vortex is not analytically integrable and hence we need to evaluate the coefficients numerically. As we have defined the coefficients in a dimensionless manner, they do not depend on the value of the vortex mass scale, M , but they do depend on the value of the fourth-order derivative term, κ (or rather the combination κM^2). We give a set of numerically evaluated coefficients for the effective Lagrangian density in Tab. II.

Using again a scaling argument [46], we can estimate the size of the sine-Gordon kink

$$\frac{1}{L} \sim \sqrt{\frac{a_{2,0,0} m_3^2}{a_{2,0,0} + \kappa(a_{2,2,0} + a_{2,0,2}) M^2 + 2c_6 a_{2,2,2} M^4}}, \quad (29)$$

where the coefficients a are functions of κM^2 , as shown in Tab. II. As examples, we can calculate the kink sizes, see Tab. III. Thus from these rough estimates, we learn that the sixth-order derivative term induces coefficients in the effective Lagrangian for the kink which increases its size (for fixed masses). The fourth-order derivative term also leads to an increase in the kink size, but to a lesser extent.

TABLE III: Rough size estimate of sine-Gordon kinks using Eq. (29) as function of the parameters of the effective theory.

$\kappa M^2 \setminus c_6 M^4$	0	1	81
0	m_3	$0.90m_3$	$0.22m_3$
1	$0.83m_3$	$0.80m_3$	$0.30m_3$
9	$0.57m_3$	$0.57m_3$	$0.44m_3$

The (full 3D) baryon charge is composed by the vortex charge and the kink charges and is given by

$$B = \frac{1}{16\pi^2} \int d^3x \frac{1}{r} \sin(f) f_r \zeta_z = Q[\zeta]_{z=z_1}^{z=z_2} = QP, \quad (30)$$

where Q is the winding number of the vortex and P is the number kinks on the string.

V. NUMERICAL SOLUTIONS

In this section, we provide explicit numerical solutions. Solutions of the baby-Skyrmion type in model 1 were already obtained in Ref. [27]. Here we will add a new lump solution living on the domain wall, which also carries baryon charge. Our other new findings are confined Skyrmions residing on the vortex string in model 2. For both cases, we need the deformation V_2 of the potentials for flat host solitons, while we do not need it for solutions possessing an S^n world-volume. On the contrary, we need no higher-derivative terms for flat host solitons, while we do need them for the solutions possessing an S^n world-volume for the stability. An exception to the rule is the baby-Skyrmion living on the domain wall, which needs both a higher-derivative term as well as the potential V_2 (but the lump on the domain wall needs neither of these).

For both models we use the relaxation method on a cubic square-lattice of size 81^3 (lattice points). We fix the boundary conditions corresponding to the host solitons as described in Sec. III and choose appropriate initial conditions for the baby-solitons in question. Then we relax the initial guess until the solution to the equations of motion is obtained with the required precision. A cross check of the solutions is the calculation of the topological baryon charge. We will now take the two models in turn.

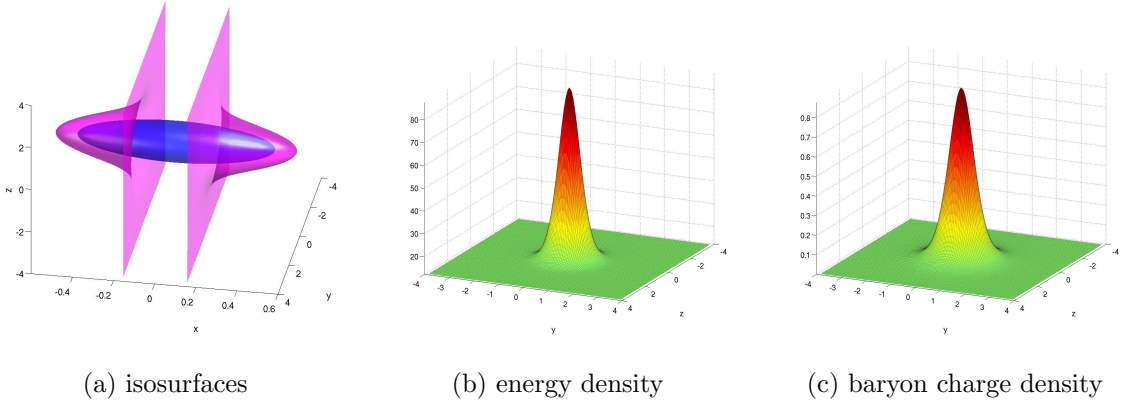


FIG. 3: The baby-Skyrmion living on the domain wall: (a) 3D view of isosurfaces for the domain wall on which a baby-Skyrmion resides; the magenta surfaces represent the energy isosurfaces at a third of the maximum of the energy and the blue surface at the center shows the baryon charge isosurface, at half its maximum value. (b) and (c) show respectively the energy density and baryon charge density at a yz -slice in the middle of the domain wall (at $x = 0$). The calculation is done on a 129^3 cubic lattice, $B^{\text{numerical}} = 0.998$ and the potential used is V_2 with $a_3 = 1$ and $M = 4, m_3 = 2$. This figure is taken from Ref. [27].

A. Model 1: Skyrmions trapped inside a domain wall

We begin with the (exact) domain wall solution of Sec. III A and add two types of baby-solitons. The first example is the baby-Skyrmion, which was obtained in [27] and we will only review it here for completeness. This solution needs both the potential V_2 as well as a higher-order derivative term. In Fig. 3 we show the baby-Skyrmion with V_2 setting $a_3 = 1$ and $\kappa = 1, c_6 = 0$ (thus only the Skyrme term is active), which is taken from [27].

The next solution, which is new, is the domain wall with a lump solution inside. This solution also carries a full unit of 3-dimensional baryon (Skyrme) charge. It is obtained for $V_2 = 0$ and no higher-derivative terms, i.e. $\kappa = 0, c_6 = 0$. The numerical solution is shown in Fig. 4. Notice that the lump has a size modulus and can thus take on any size. We also do not capture the full baryon charge because we resolve only the center of the lump with the finite lattice points. This is not a problem of the solution but of the lattice size. A larger lattice will capture more of the baryon charge (or alternatively a small lump on the same lattice with a worse resolution).

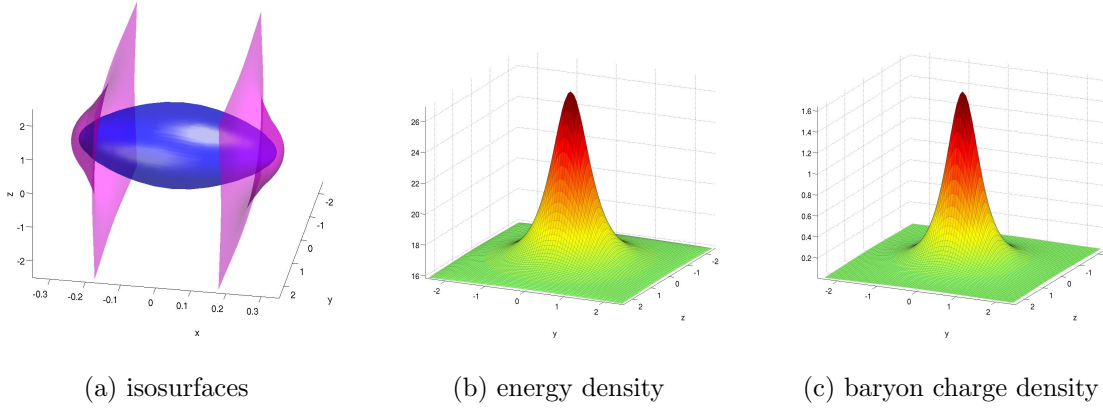


FIG. 4: The domain wall with a trapped Skymion in the theory without higher-derivative and potential terms. (a) 3D view of isosurfaces for the energy density and baryon charge density as in Fig. 3. (b) and (c) show respectively the energy density and baryon charge density at a yz -slice in the middle of the domain wall (at $x = 0$). The calculation is done on an 81^3 cubic lattice, with $M = 4$ and $B^{\text{numerical}} = 0.877$.

B. Model 2: Skymions confined by vortices

In this section we take the vortex solution of Sec. IIIB and add a sine-Gordon kink on its world-volume. As already mentioned, for the straight vortex we need a finite potential V_2 for the baby-soliton. If we choose a linear potential, i.e. $a_3 = 1$, the kink on the vortex corresponds to a full unit of baryon charge, while for the quadratic potential, i.e. $a_3 = 2$, each kink corresponds to half a unit of baryon charge.

The straight vortex possesses sine-Gordon kinks if the potential V_2 is turned on even without higher-derivative terms. The presence of the Skyrme term (the fourth-order derivative term) widens both the vortex itself and the kink living on the vortex (see Sec. IV B, whereas the sixth-order derivative term does not alter the vortex solution, but it does widen the kink – even more than the Skyrme term, see Tab. III.

First we will present the vortex with a full sine-Gordon kink living on its world-volume in the case of no higher-derivative terms, see Fig. 5. This kink was made with kink mass $m_3 = 0.22$, vortex mass $M = 3$ and the kink length was measured as

$$L_{\text{kink}} = \sqrt{\frac{\int d^3x \, z^2 \mathcal{E}_{\text{kink}}}{\int d^3x \, \mathcal{E}_{\text{kink}}}} \sim 3.35, \quad (31)$$

which one can compare with the analytic formula $\frac{\pi}{2\sqrt{3}m_3} \simeq 4.14$. The reason for the smaller value in the numerical result is due to the binding energy present on both sides of the center of the kink, see Fig. 5d.

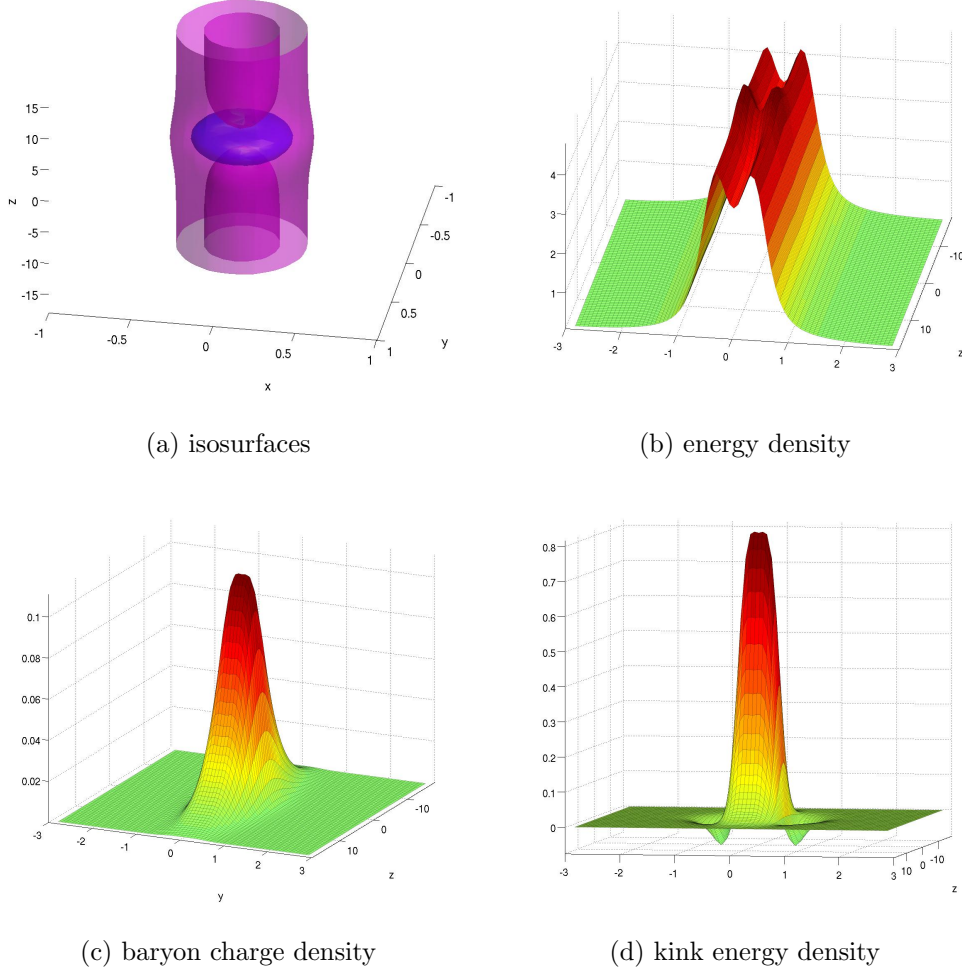


FIG. 5: The vortex with a trapped Skymion which is manifested as a sine-Gordon kink on its world-volume, in the theory with no higher-derivative terms. (a) 3D view of isosurfaces for the energy density and baryon charge density. (b) and (c) show respectively the energy density and baryon charge density at a yz -slice through the vortex (at $x = 0$). (d) shows the energy of the kink (which is the total energy with the vortex energy subtracted off). Notice the negative dips on each side of the peak in the kink energy; we interpret those as binding energy. The calculation is done on an 81^3 cubic lattice, with $M = 3$, $m_3 = 0.22$ and the baryon charge is $B^{\text{numerical}} = 0.991$.

Now we will consider a different example, namely the full sine-Gordon kink in the vortex

theory with the sixth-order derivative term turned on ($c_6 = 1$) while the Skyrme term is still off ($\kappa = 0$). As already mentioned, it does not alter the vortex solution far away from the kink, but it does increase the length of the kink living on its world-volume. In Fig. 6 we present this solution and we have taken the kink mass to be $m_3 = 1$ and the vortex mass $M = 3$. The kink length was measured with Eq. (31) to be $L_{\text{kink}} \sim 2.44$. The effective theory predicted this kink to be almost five times longer than that without the sixth-order derivative term; whereas numerically it is only about 2.7 times longer (according to this measure); but recall that the scaling argument is just a rough estimate neglecting the actual integrals (or rather assuming them to be of order one). We chose the kink mass in the latter vortex solution (in Fig. 6) to be $m_3 = 0.22$ such that the kink should have approximately the same length as that with a sixth-order derivative term with $m_3 = 1$. Measuring this ratio numerically we get ~ 0.73 which on the one hand determines the accuracy of the estimate; but also confirms that the effective theory is qualitatively correct.

The last example we will consider, is the vortex compactified on a circle, S^1 , without a potential V_2 , which has a free theory living on its world-volume. Existence of this solution without angular momentum, requires a higher-derivative term; here we will use only the sixth-order derivative term. In Fig. 7 is shown a numerical solution with vortex mass $M = 4$ and $c_6 = 1$. Notice that the energy density is torus-like to some extent (it has a valley almost halfway down in energy density) but the baryon charge density remains as a ball-like object with a dip in the energy density at the origin.

VI. SUMMARY AND DISCUSSION

In this paper we have exhausted the possibilities (known so far) of Skyrmions in different disguises. By trapping a Skyrmion on a domain wall, it hosts a baby-Skyrmion while the full system has a 3-dimensional Skyrme (baryon) charge. If the domain wall is compactified it is again a normal Skyrmion, but having its energy distributed as in a ball-like object. Using the parameter space of the model, it is possible to obtain a spherical shell-like object – by for instance having very large sixth-order derivative term, see [28]. In this paper, we find the new and last piece of the puzzle, i.e. the Skyrmion trapped on a vortex string, which looks like a sine-Gordon kink on the vortex world-sheet. We find the existence of this object by an effective theory approach [34] and by explicit numerical calculations. The last object

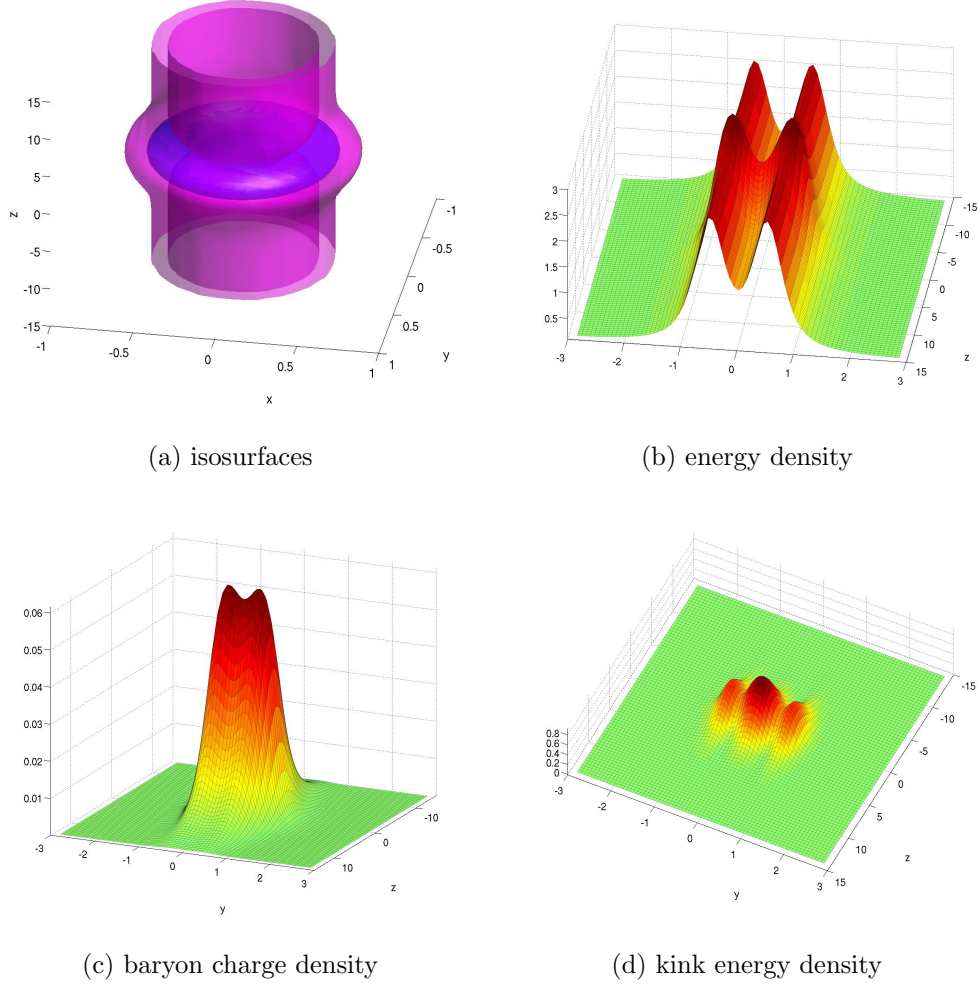


FIG. 6: The vortex with a trapped Skymion which is manifested as a sine-Gordon kink on its world-volume, in the theory with a sixth-order derivative term. (a) 3D view of isosurfaces for the energy density and baryon charge density. (b) and (c) show respectively the energy density and baryon charge density at a yz -slice through the vortex (at $x = 0$). (d) shows the energy of the kink (which is the total energy with the vortex energy subtracted off). Notice that instead of binding energy, the higher-derivative term induces sub-peaks on the side of the kink. The calculation is done on an 81^3 cubic lattice, with $M = 3$, $m_3 = 1$ and the baryon charge is $B^{\text{numerical}} = 0.990$.

we find here is the vortex compactified on a circle, which thus carries a Skyrme charge by having a twist on its modulus. Kinks can furthermore live on this torus-like object, but we leave such studies for future developments.

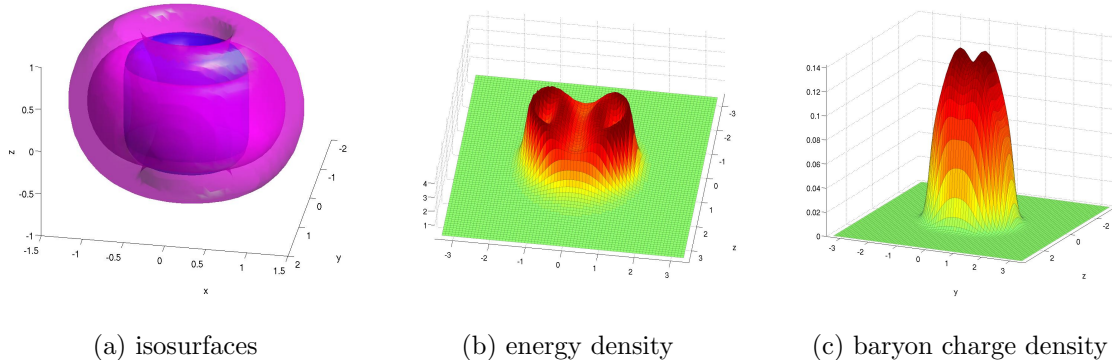


FIG. 7: The vortex compactified on a circle in the theory with a sixth-order derivative term and without the potential V_2 . (a) 3D view of isosurfaces for the energy density and baryon charge density as in Fig. 3. (b) and (c) show respectively the energy density and baryon charge density at a yz -slice (at $x = 0$). The calculation is done on an 81^3 cubic lattice, with $M = 4$ and $B^{\text{numerical}} = 0.99990$.

Let us comment on the accuracy of the comparison between the lengths predicted from the effective theory and the actual numerical calculation we carried out. As emphasized both here and in [34], the effective theory relies heavily on the separation of scales when taking only the leading-order contribution into account. Higher-order corrections have not yet been calculated explicitly, although it is straightforward. However, the numerical solutions are all done on a finite square-lattice which makes a too large separation of scale inconvenient, i.e. memory and run-time consuming, which is why we have only an order 3–4 between the mass scales in the systems studied.

We have constructed a single sine-Gordon kink residing in a vortex in this paper, however, it is also possible to make a sine-Gordon kink crystal, which is described by the elliptic function $\text{sn}(x)$ [52].

In Tab. I, we have summarized the topological incarnations of lumps (baby-Skyrmions, or sigma-model instantons), Skyrmions, and Yang-Mills instantons. There seems to be certain relations among the homotopy in the bulk (resultant solitons), the homotopy of host solitons, and the homotopy of world-volume solitons, but an exact mathematical correspondence is yet to be clarified. We can do the same for Hopfions (knot solitons) [48]; Hopfions can be realized as sine-Gordon kinks on a toroidal domain wall [49].

The BEC Skyrme model which we consider in this paper also admits D-brane solitons [50], that is, vortices ending on a domain wall, since the corresponding BECs admit them [31, 33]. Therefore, this model admits various solitons with various codimensions: domain walls, vortices and Skyrmions, and their composites. The dynamics of these solitons remain as an interesting problem to explore. For instance, Skyrmions were proposed to be created after the annihilation of a brane and anti-brane [31, 51].

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Appendix A: Two-component Bose-Einstein condensates

Two-component Bose-Einstein condensates with two wave functions ϕ_1, ϕ_2 have the potential term [29]

$$V = -\mu_1|\phi_1|^2 - \mu_2|\phi_2|^2 + \frac{g_{11}}{2}|\phi_1|^4 + \frac{g_{22}}{2}|\phi_2|^4 + g_{12}|\phi_1|^2|\phi_2|^2. \quad (\text{A1})$$

When we consider the case $g_{11} = g_{22} = g$ and $\mu_1 = \mu_2 = \mu$, the potential reads

$$\begin{aligned} V &= -\mu|\phi_1|^2 - \mu|\phi_2|^2 + \frac{g}{2}|\phi_1|^4 + \frac{g}{2}|\phi_2|^4 + g_{12}|\phi_1|^2|\phi_2|^2 \\ &= \frac{g}{2}(|\phi_1|^2 + |\phi_2|^2 - v^2)^2 + m^2|\phi_1|^2|\phi_2|^2 + \text{const}. \end{aligned} \quad (\text{A2})$$

with

$$v^2 \equiv \frac{\mu}{g}, \quad \frac{1}{2}m^2 \equiv g_{12} - g. \quad (\text{A3})$$

We consider the strong coupling limit

$$g, g_{12} \rightarrow \infty, \quad m^2 = \text{fixed}, \quad (\text{A4})$$

which yields the $O(4)$ nonlinear sigma model with the constraint $|\phi_1|^2 + |\phi_2|^2 = v^2$, whose target space is $S^3 \simeq SU(2)$. The potential is

$$V = \frac{1}{2}m^2|\phi_1|^2|\phi_2|^2 + \text{const.} \quad (\text{A5})$$

which is of the form considered in this paper.

Appendix B: Half a Skyrmion trapped on a vortex

As promised earlier in the paper, half a Skyrmion can be manifested by making a sine-Gordon kink with the potential V_2 and setting $a_3 = 2$, such that only “half” a twist of the $U(1)$ modulus is needed, resulting in a Skyrmion with half a unit of baryon charge. The solution is shown for the vortex theory without higher-derivative terms, vortex mass $M = 3$, kink mass $m_3 = 0.22$ and is shown in Fig. 8. As can be seen from the figures, the half-Skyrmion is qualitatively the same as the full Skyrmion, except for possessing only half the baryon charge.

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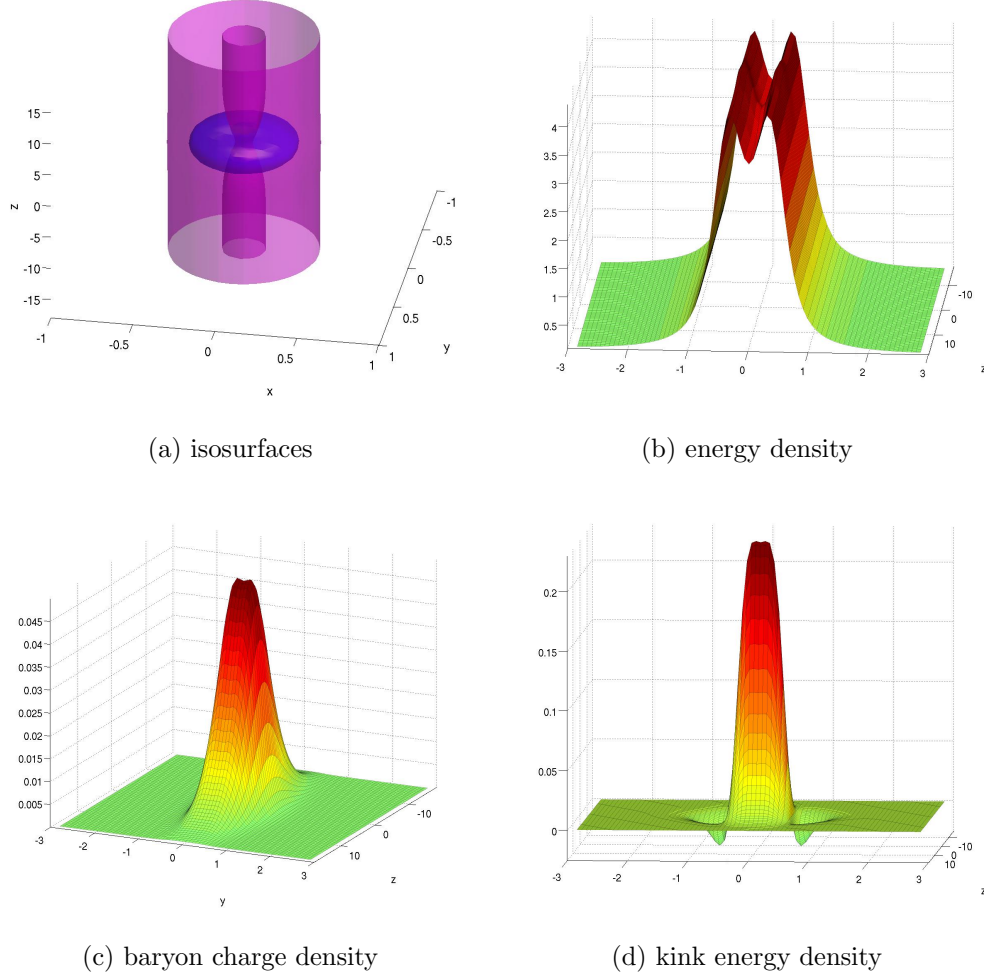


FIG. 8: The vortex with a trapped half-Skyrmion which is manifested as “half” a sine-Gordon kink on its world-volume, in the theory with no higher-derivative terms. (a) 3D view of isosurfaces for the energy density and baryon charge density. (b) and (c) show respectively the energy density and baryon charge density at a yz -slice through the vortex (at $x = 0$). (d) shows the energy of the kink (which is the total energy with the vortex energy subtracted off). Notice the negative dips on each side of the peak in the kink energy; we interpret those as binding energy. The calculation is done on an 81^3 cubic lattice, with $M = 3$, $m_3 = 0.22$ and the baryon charge is $B^{\text{numerical}} = 0.497$.

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